Classification of possible finite-time singularities by functional renormalization

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Starting from a representation of the early time evolution of a dynamical system in terms of the polynomial expression of some observable $\phi(t)$ as a function of the time variable in some interval $0 \leqslant t \leqslant T$, we investigate how to extrapolate/forecast in some optimal stability sense the future evolution of $\phi(t)$ for time $t > T$. Using the functional renormalization of Yukalov and Gluzman, we offer a general classification of the possible regimes that can be defined based on the sole knowledge of the coefficients of a second-order polynomial representation of the dynamics. In particular, we investigate the conditions for the occurrence of finite-time singularities from the structure of the time series, and quantify the critical time and the functional nature of the singularity when present. We also describe the regimes when a smooth extremum replaces the singularity and determine its position and amplitude. This extends previous works by (1) quantifying the stability of the functional renormalization method more accurately, (2) introducing more global constraints in terms of moments, and (3) going beyond the "mean-field" approximation.

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I. INTRODUCTION

Finite-time singularities in the dynamical equations used to describe natural systems are not always pathologies that should be thrown away or ignored, but may often betray important useful information on the characteristic properties of the real system. Actually, spontaneous singularities in ordinary and partial differential equations are quite common and have been found in many well-established models of natural systems, either at special points in space such as in the Euler equations of inviscid fluids [1,2], in the surface curvature on the free surface of a conducting fluid in an electric field [3], in vortex collapse of systems of point vortices [4], in the equations of general relativity coupled to a mass field leading to the formation of black holes [5], in models of micro-organisms aggregating to form fruiting bodies [6], or in the more prosaic rotating coin (Euler's disk) [7,8]. Some more complex examples are models of rupture and material failure [9–11], earthquakes [12,13] and stock market crashes [14,15].

In a recent work [16], we have developed theoretical formulas for the prediction of the singular time of systems which are *a priori* known to exhibit a critical behavior, based solely on the knowledge of the early time evolution of an observable. From the parametrization of such early time evolution in terms of a low-order polynomial of the time variable, the functional renormalization approach introduced by Yukalov and Gluzman allows one to transform this polynomial into a function that is asymptotically a power law. The value of the critical time t_c, *conditioned* on the assumption that t_c exists, can then be determined from the knowledge of the coefficients of the polynomials. Reference [16] has tested with success this prediction scheme on a specific example and showed that this approach gives more precise and reliable predictions than through the use of the exact real power law model. This is a rather surprising and paradoxical observation in contradiction with common wisdom according to which the best and most reliable prediction should be obtained from the use of the exact underlying model of the dynamical behavior [17]. The reason why this is not always true is that our method has shown that approximate solutions can be more stable and more reliable when devised so as to maximize the criterion of stability with respect to perturbations in the functional space [16].

Here, we extend this work by offering a general classification of the possible regimes that can be defined based on the sole knowledge of the coefficients of the polynomial expansion of some observable as a function of the time variable. We mostly restrict our analysis to second-order polynomials. As a consequence, the classification can be organized in a unique way in terms of the single signed Froude parameter, ratio of the square of the velocity to the acceleration.

Let us assume that the dynamical behavior of a system is sampled to obtain a time series $\phi(t)$ in the interval $0 \leqslant t \leqslant T$. The question we address in this work is how to extrapolate/forecast in some optimal sense the future evolution of $\phi(t)$ for time $t > T$. Beyond a simple extrapolation, we ask whether it is possible to detect the germs of a finite-time singularity from the structure of the time series, and quantify the critical time and the functional nature of the singularity when present. We also aim at classifying the regimes when a smooth extremum replaces the singularity and at determining its position and amplitude.

We shall work in the framework of the functional renormalization method, which constructs the extrapolation for $t > T$ from a resummation of the time series represented by a simple polynomial expansion in powers of time t, where t is counted from the beginning of the recorded time series.

II. SUMMARY OF THE FUNCTIONAL RENORMALIZATION APPROACH TO EXTRAPOLATION OF TIMES SERIES

The mathematical foundation of the functional renormalization approach used here can be found in earlier publications [18–30], which we summarize briefly. Let perturbation theory (or some fitting procedure) give for the time series $\phi(t)$ the succession of nonrandom approximations $\phi_n(t)$ where $n=0,1,2,\ldots$ enumerates the order of the increasingly precise approximations. The case of a polynomial expansion will be discussed below,

$$\phi_n(t)=\sum_{k=0}^{n} a_k t^k, \quad n=0,1,2,3,\ldots. \tag{1}$$

This expansion (1) is, in principle, defined for t sufficiently small and based on information about the time series up to the time T. The expansion may have no meaning if continued straightforwardly to the region of finite $t>T$. The problem of reconstructing the value of the function in some distant moment of time from the knowledge of its asymptotic expansion as $t\to 0$ is called a renormalization or resummation problem in theoretical physics. An analytical tool for the solution of this problem, called algebraic self-similar renormalization, has been developed recently [18–30] of which we summarize the salient points useful for the present work.

It is convenient to remove the constant term and consider the time series $p(t)\equiv\phi(t)-\phi_0$ represented by the sequence of polynomial approximations $p_i(t), i=1,2,\ldots,n$,

$$p_1(t)=a_1 t, \quad p_2(t)=p_1(t)+a_2t^2, \ldots,$$

$$p_n(t)=p_{n-1}(t)+a_n t^n. \tag{2}$$

The algebraic self-similar renormalization starts by applying to the approximations (2) a simple algebraic multiplication, thus defining a new sequence, $P_i(t,s)=t^s p_i(t)$, $i=1,2,\ldots,n$, with $s\geqslant 0$. This transformation increases the powers of the approximation sequences (1) and (2). This formal manipulation effectively increases the order of the expansion and provides a trick to effectively taking into consideration more points from the system trajectory. In the first part of the paper, we use the strongest form of this transformation corresponding to formally taking the limit $s\to\infty$. This will be shown to provide a representation of the renormalized function as an embedded set of exponentials associated with a singularity $\phi\sim(t_c-t)^z$, with critical exponent z imposed to have an absolute value equal to 1. This limit can thus be considered as analogous to a "mean-field" regime with the mean-field value of the critical exponent $z=-1$. This can, for instance, represent the exponent of the derivative of a quantity exhibiting a weak logarithmic divergence. In the second part of the paper, we shall relax this limit and determine both t_c and the index z self-consistently.

The second step consists in considering the sequence of transformed approximations $P_i(t,s)$, as a dynamical system in the discrete "order time" equal to the order $i=0,1,\ldots,n-1$ of the approximation. In order to keep the information on the system evolution with the real time t, it is convenient to introduce a new variable φ and define the so-called expansion function $t(\varphi,s)$ from the equation $P_1(t,s)=a_1t^{1+s}=\varphi$, which gives $t(\varphi,s)=(\varphi/a_1)^{1/(1+s)}$. We then construct the discrete flow in the space of approximations indexed by the "order time" as

$$y_i(\varphi,s)\equiv P_i(t(\varphi,s),s). \tag{3}$$

One can then write the equation of evolution in the space of approximations as a function of the discrete "order time" in the form of the functional self-similarity relation

$$y_{i+p}(\varphi,s)=y_i(y_p(\varphi,s),s). \tag{4}$$

Expression (4) provides the necessary condition for the self-consistency of the cascade of discrete approximations and ensures the convergence of P_i's. Expression (4) is nothing but the application of the Caccioppoli-Banach principle for the existence of a stable fixed point (see, for instance, Chap. XVI of Ref. [31]).

At this stage, the efficiency of the algebraic transformation can be checked by analyzing its stability. This is done for the sequence of $P_i(t,s)$'s by calculating the so-called local multipliers (essentially proportional to the exponential of Lyapunov exponents)

$$m_i(t,s)\equiv\left[\frac{\partial y_i(\varphi,s)}{\partial\varphi}\right]_{\varphi=P_1(t,s)}. \tag{5}$$

When all $|m_i(t,s)|<1$, the convergence of the sequence P_i is guaranteed. To implement concretely the calculations, we use the integral form of the self-similarity relation (4)

$$\int_{P_{i-1}}^{P_i^*}\frac{d\varphi}{v_i(\varphi,s)}=\tau, \tag{6}$$

where the cascade velocity is $v_i(\varphi,s)=y_i(\varphi,s)-y_{i-1}(\varphi,s)$ and τ is the minimal number of steps of the approximation procedure needed to reach the fixed point $P_i^*(t,s)$ of the approximation cascade. It is possible to find $P_i^*(t,s)$ explicitly and to perform an inverse algebraic transform after which the limit $s\to\infty$ is to be taken. This completes the first loop of the self-similar renormalization. This procedure can be repeated as many times as it is necessary to renormalize all polynomials that appear at the preceding steps. This is the main idea of the self-similar bootstrap [22].

Completing this program, we come to the following sequence of self-similar exponential approximants:

$$p_j^*(t,\tau_1,\tau_2,\ldots,\tau_{j-1})$$
$$=a_1t\exp\left[\frac{a_2}{a_1}t\tau_1\cdots\exp\left(\frac{a_j}{a_{j-1}}\tau_{j-1}t\right)\right],$$
$$j=2,3,\ldots,n \tag{7}$$

and

$$\phi_j^*(t,\tau_1,\tau_2,\ldots,\tau_{j-1})=p_j^*(t,\tau_1,\tau_2,\ldots,\tau_{j-1})+a_0. \tag{8}$$

Explicitly, for the three first orders, we obtain

$$\phi_2^*(t,\tau_1)=a_1t\exp\left(\frac{a_2}{a_1}t\tau_1\right)+a_0, \tag{9}$$

$$\phi_3^*(t,\tau_1,\tau_2)=a_1 t \exp\left[\frac{a_2}{a_1} t\tau_1 \exp\left(\frac{a_3}{a_2} t\tau_2\right)\right]+a_0, \quad (10)$$

$$\phi_4^*(t,\tau_1,\tau_2,\tau_3)=a_1 t \exp\left\{\frac{a_2}{a_1} t\tau_1 \exp\left[\frac{a_3}{a_2} t\tau_2 \exp\left(\frac{a_4}{a_3} t\tau_3\right)\right]\right\} + a_0. \quad (11)$$

Compared to previous works [18–30], we introduce the following innovation to improve on the selection of the stable extrapolations (or scenarios). In order to check whether the sequence of $\phi_j^*(t,\tau_1,\tau_2,\ldots,\tau_{j-1})$ really converges, in addition to calculating the multipliers $m_i(t,s)$ defined by Eq. (5), we analyze the multipliers $M_j(t,\tau_1,\tau_2,\ldots,\tau_{j-1})$ corresponding specifically to $\phi_j^*(t,\tau_1,\tau_2,\ldots,\tau_{j-1})$. For this, we construct again an approximation cascade as described above and define

$$M_j(t,\tau_1,\tau_2,\ldots,\tau_{j-1}) \equiv \left[\frac{\partial p_j^*(\varphi,\tau_1,\tau_2,\ldots,\tau_{j-1})}{\partial\varphi}\right]_{\varphi=P_1(t,0)}. \quad (12)$$

Such a definition of multipliers allows us to compare the convergence of the expansion and of the renormalized expansion, making clear the improvements that can be expected *a priori* from the technique. If the values of M_j are systematically smaller than the corresponding m_j in the region of $t\leqslant T$ and the overall convergence properties are improved, one can expect that the renormalized expressions will work better than the original expansion at $t>T$.

The final step consists in determining the control parameters $\tau_1,\tau_2,\ldots,\tau_{j-1}$, by expanding $\phi_j^*(t,\tau_1,\tau_2,\ldots,\tau_{j-1})$ in the vicinity of $t=0$, and requiring that this expansion agrees term by term with the initial one $\phi_j(t)$ given by Eq. (1). This corresponds to imposing the condition of "accuracy through order," which ensures that the approximants provide a systematic approximation scheme: if the index j is increased, the accuracy of the approximant to the expansion in the vicinity of the expansion point $t=0$ also increases. This standard property is required in the construction, for instance, of the method of Padé approximations [36] of the expansion (1).

The accuracy-through-order relationship, which holds for the $\phi_n^*(t)$ approximant, means that $\phi(t)-\phi_n^*(t)\simeq O(t^{1+n})$, $t\to 0$. Technically, in order to obtain the corresponding control parameters, we expand the approximant near the origin $t\to 0$ and equate the coefficients of the expansion to the corresponding coefficients of the polynomial expression (1). For example, for $n=2$ and $n=3$, in the cases of the two simplest approximants, we obtain as $t\to 0$,

$$\phi_2^*(t,\tau_1)\simeq a_0+a_1 t+a_2\tau_1 t^2+O(t^3), \quad (13)$$

$$\phi_3^*(t,\tau_1,\tau_2)\simeq a_0+a_1 t+a_2\tau_1 t^2+\frac{\tau_1(2a_3a_1\tau_2+a_2^2\tau_1)}{2a_1} + O(t^4), \quad (14)$$

and we easily find that $\tau_1=1$ and $\tau_2=1-a_2^2/(2a_1a_3)$.

The problem of finding the control parameters is not too complicated, since we are dealing formally with a nonlinear but rather simple algebraic systems for τ's, i.e., the knowledge of the lowest-order τ's automatically gives us the expression for the τ in the next order. Using the anlytical package MATHCAD 8, we have performed calculations up to the tenth order of the perturbative expansion, without encountering any serious diffuculties, except for quite lengthy analytical expressions. However, for very large orders j, some special procedure, possibly combining numerical and analytical approaches should be developed.

For each j, we obtain $j-1$ self-similar approximants for the sought function (where all control parameters τ are now known functions of the parameters a, with $\tau_1\equiv 1$)

$$\phi_{j1}^*(t,1,1,\ldots,1),$$

$$\phi_{j2}^*(t,1,\tau_2,1,\ldots,1), \quad (15)$$

$$\phi_{jj-1}^*(t,1,\tau_2,\ldots,\tau_{j-1}), \quad (16)$$

which differ according to the number of control functions being used. We can now construct a table of self-similar approximants, varying both j and the number of controls. Accordingly, we can define the table of multipliers, varying both j and the number of controls. For instance, for $j=4$, we have the following table of approximations:

$$\phi_{21}^*(t)=\phi_j^*(t,1),$$

$$\phi_{31}^*(t)=\phi_3^*(t,1,1), \quad \phi_{32}^*(t)=\phi_3^*(t,1,\tau_{2,}),$$

$$\phi_{41}^*(t)=\phi_4^*(t,1,1,1), \quad \phi_{42}^*(t)=\phi_4^*(t,1,\tau_2,1),$$

$$\phi_{43}^*(t)=\phi_4^*(t,1,\tau_2,\tau_3).$$

For all approximants other than $\varphi_{n,n-1}^*(t)$, the accuracy-through-order relationship holds in a relaxed form, e.g., $\phi(t)-\phi_{nn-2}^*(t)\simeq O(t^n)$, $t\to 0$. In this case, we proceed as described above but set the highest-order control parameter equal to 1, while the other control parameters are determined from the expansion.

We stress that suppressing (allowing) an error in the limit of $t\to 0$ does not necessarily mean that the error will be minimized (increased) for the sought "nonsmall" values of t. Sometimes, allowing a larger but still rather small error at vanishingly small t's may help in decreasing the error in the critical region, since the trajectory described by the corresponding approximant may become more stable. The approximant that respects the standard form of the accuracy-through-order relationship appears to be the best in the "cosine" example presented below. But this is not necessarily the general case, as is demonstrated by the other examples also discussed below.

In order to select the scenarios for the extrapolation to the future from the initial time series, one needs to examine both the properties of convergence of the sequences of multipliers M, and of approximants ϕ^*. The best scenario is the one that exhibits the best simultaneous convergence of both sequences. The resulting limiting fixed point read from the table of approximants should then be taken for the sought extrapolation function. If there are more than one limiting points, the sought function should be constructed by taking their average with weights inversely proportional to the multipliers (see Ref. [30]). We now illustrate this general methodology on specific examples for which a finite-time singularity might exist.

III. CLASSIFICATION OF SINGULAR AND NONSINGULAR BEHAVIORS BASED ON SECOND-ORDER EXPANSIONS

A. Definitions

Let us now consider the simplest nontrivial case allowing for the possible existence of a finite-time singularity, namely, a second-order polynomial representation

$$\phi_2(t) \simeq 1 + a_1 t + a_2 t^2 \quad (t \to 0) \tag{17}$$

of the initial time series. In the language of Ref. [32], a_1 is the velocity and $2a_2$ is the acceleration. The relative influence of the velocity and acceleration is quantified by the so-called Froude number [32] defined by

$$F_d \equiv \frac{a_1^2}{a_2}. \tag{18}$$

Working with the "direct" series (17), the program described in the preceding section provides the sought observable, which we name $\Phi_2^*(t)$, equal to the resummed approximant,

$$\Phi_2^*(t) = \phi_{21}^*(t) = 1 + a_1 t \exp\left[\frac{a_2}{a_1} t\right]. \tag{19}$$

We will also study, when necessary, the inverse function $\phi_{2I}(t)$ of $\phi_2(t)$,

$$\phi_{2I}(t) = 1/\phi_2(t) = 1 - a_1 t + (-a_2 + a_1^2) t^2 + \cdots \equiv 1 + b_1 t + b_2 t^2 + \cdots. \tag{20}$$

The corresponding Froude number F_I is

$$F_I \equiv \frac{b_1^2}{b_2} = \frac{a_1^2}{a_1^2 - a_2} = \frac{F_d}{F_d - 1} \left(F_d = \frac{F_I}{F_I - 1} \right). \tag{21}$$

In cases when better convergence can be achieved by working with the "inverse" series (20), the observable $\Phi_2^*(t)$ will be expressed through the approximant corresponding to the inverse series as follows:

$$\Phi_2^*(t) = [\phi_{21I}^*(t)]^{-1} = \left(1 + b_1 t \exp\left[\frac{b_2}{b_1} t\right] \right)^{-1}. \tag{22}$$

The convergence of our procedure is checked by estimating the value of the multiplier

$$M_{21}(t) = \left(1 + \frac{a_2}{a_1} t \right) \exp\left(\frac{a_2}{a_1} t \right). \tag{23}$$

The same expression gives the multiplier for the inversion function by changing all the coefficients a_i's into b_i's, where the b_i's are related to the a_i's via expression (20).

B. Negative velocity $a_1 < 0$ (downward trend)

1. Negative acceleration $a_2 < 0$

Working with positive observables, this case corresponds to the possibility that the observable vanishes in finite time. This zero-crossing occurs at t_{c21} given by

$$1 + F_d Z_d \exp[Z_d] = 0, \tag{24}$$

where $Z_d = (a_2/a_1) t_{c21}$. The corresponding multiplier $M_{21}(t_{c21})$ is larger than 1, signaling a possible problem with the convergence of the functional renormalization method. This may signal an instability in the time dynamics of the time series close to the zero-crossing time.

2. Moderate positive acceleration $0 < a_2 < a_1^2/e$

The observable goes again to zero in finite time. The time t_{c21} at which the observable vanishes is given again by Eq. (24). The corresponding multiplier $M_1(t_{c21})$ is now less than 1, signaling a stable scenario.

3. Strong positive acceleration $a_1^2/e < a_2$

The acceleration is sufficiently positive to counterbalance the negative trend and the observable goes through a minimum before rebounding upward. The time $t_{\min}$ at which the minimum $\Phi_{\min}^* = 1 - F_d/e$ is reached is given by

$$t_{\min} = -\frac{a_1}{a_2}. \tag{25}$$

This is a very stable situation since the multiplier is zero at the minimum. Our analysis suggests that the extrapolation to the future is the most credible for this situation in the downward trend case.

C. Positive velocity $a_1 > 0$ (upward trend)

1. Negative acceleration: $a_2 < 0$

The observable increases up to a maximum $\Phi_{\max}^* = 1 - F_d/e$ and then decreases beyond it. The time $t_{\max}$ of the maximum is given by

$$t_{\max} = -\frac{a_1}{a_2}.$$

This is a very stable situation since the multiplier M_{21} is less than 1 for arbitrary time t and is zero at the maximum.

2. Positive acceleration

This case requires the inversion of the time series, since the multipliers of the direct series are always larger than 1:

the expansion is not convergent and does not supply any information on the real-axis singularities. The inversion maps these two cases on the cases previously considered with multipliers smaller than 1. In addition, the corresponding series has alternating signs, suggesting that a finite-time singularity is indeed present.

3. Moderate positive acceleration: $0<a_2<(e-1)/ea_1^2$

The observable increases up to a maximum and then decreases beyond it. The time $t_{\max}$ of the maximum is given by

$$t_{\max}=-\frac{b_1}{b_2}=\frac{a_1}{a_1^2-a_2}, \tag{26}$$

and the value at the maximum is

$$\Phi_{\max}=\left(1-\frac{F_I}{e}\right)^{-1}. \tag{27}$$

This solution is very stable and thus very credible as the multiplier for $\phi_{21I}^*(t)$ is always smaller than 1 and vanishes at the maximum. The original function is obtained by the inversion $\Phi_2^*(t)=[\phi_{21I}^*(t)]^{-1}$ [see also Eq. (20)].

4. Large positive acceleration: $(e-1)/ea_1^2<a_2$ or $F_d<e/(e-1)$

There is an upward finite-time singularity at t_{c21} given by the zero of the inverted renormalized expansion, i.e., by the solution of

$$\phi_{21I}^*(t_{c21})=1+F_iZ_i\exp[Z_i]=0, \tag{28}$$

where $Z_i=t_{c21}(a_2-a_1^2)/a_1$. The multiplier for $\phi_{21I}^*(t)$ is smaller than 1, signaling the convergence of the theoretical procedure for the inverse functions. Recall again that, in order to return to the original function, the inversion $\Phi_2^*(t)=[\phi_{21I}^*(t)]^{-1}$ should be performed, leading to a divergence of the original observable at the point where its inverse crosses zero.

D. Example

Let us consider a function with known singular behavior and compare the results obtained by means of self-similar approximations with the exact values. We consider the function $1/\cos(t)$ which was found [33] to describe the time evolution of the crack length of a self-consistent model of damage and used [16] for prediction tests. Starting from $t=0$, the function $1/\cos(t)$ possesses a singularity at $t=\pi/2\approx1.5708$, and its expansion up to second order in powers of t^2 reads

$$\phi_2(t)\simeq1+1/2t^2+5/24t^4+\cdots. \tag{29}$$

Note that, since $1/\cos(t)$ is an even function, the relevant variable for the expansion is indeed t^2 and our previous classification must be applied to the expansion (29) up to second order in t^2 (i.e., to fourth-order t^4).

The corresponding Froude number is $F_d=1.2$ and obeys the condition $1<F_d<e/(e-1)$, i.e., the preceding section shows that one can anticipate a finite-time singularity, only on the basis of the quadratic polynomial expansion. The expansion of the inverse of $\phi_2(t)$ reads

$$\phi_{2I}(t)\simeq1-1/2t^2+1/24t^4+\cdots. \tag{30}$$

As expected from our previous analysis in terms of the stability of the renormalization flow quantified by the multipliers, the position of the critical time is better estimated from the zero of $\phi_{21I}^*(t_{c21})=0$, which gives $t_{c21}=1.566\,45$, compared to the value $t_{20}=1.592\,45$ suggested from the condition $\phi_{20I}(t_{20})=0$. This last value t_{20} is nothing but the prediction of the critical time from the ‘‘bare’’ polynomial expansion. We refer to Eq. (16) for the definition of $\phi_{21I}^*(t)$ and $\phi_{20I}(t)$. This shows that using a control parameter, which ensures that the renormalized expansion retrieves the polynomial expansion, improves the prediction of the critical time.

IV. ILLUSTRATION OF THE SELECTION OF SCENARIOS BY THE CONVERGENCE OF HIGHER-ORDER APPROXIMANTS

We now exploit the previous example of the function $1/\cos(t)$ described in Sec. III D to illustrate how the concept of convergence of the approximants developed in Sec. II can be used for an improved determination of the singularity. As we pointed out in Sec. II, the approximant that respects the standard form of the accuracy-through-order relationship appears to be the best in the ‘‘cosine’’ example. In this section, we also present two other models in which this is not the case, suggesting the usefulness of our procedure based on the minimization of multipliers.

A. The cosine model

Higher-order expansions of the inverse of the function $1/\cos(t)$ are given by

$$\phi_{3I}(t)\simeq\phi_{2I}(t)-1/720t^6+\ \ldots\qquad(b_3=-1/720) \tag{31}$$

and

$$\phi_{4I}(t)\simeq\phi_{3I}(t)+1/40320t^8+\ \ldots\qquad(b_4=1/403\,20). \tag{32}$$

The corresponding higher-order approximants are

$$\Phi_3^*(t)=(\phi_{3I}^*(t))^{-1}=\left\{1+b_1t^2\exp\left[\frac{b_2}{b_1}t^2\exp\left(\frac{b_3}{b_2}\tau_2t^2\right)\right]\right\}^{-1}, \tag{33}$$

with

$$\tau_2=1-\frac{b_2^2}{2b_1b_3}, \tag{34}$$

and

$$\Phi_4^*(t)=[\phi_{4I}^*(t)]^{-1}$$
$$=\left(1+b_1t^2\exp\left\{\frac{b_2}{b_1}t^2\exp\left[\frac{b_3}{b_2}\tau_2t^2\right.\right.\right.$$
$$\left.\left.\left.\times\exp\left(\frac{b_4}{b_3}\tau_3t^2\right)\right]\right\}\right)^{-1}, \tag{35}$$

with

$$\tau_3=-\frac{b_3}{12b_1b_2b_4(b_2^2-2b_1b_3)}(24b_1^2b_4b_2+5b_2^4-12b_1^2b_3^2$$
$$-12b_1b_3b_2^2). \tag{36}$$

As explained in Sec. II, the control parameters τ_2 and τ_3 are determined from the condition that the expansion of the approximants at small t coincide with the perturbative expression of the initial polynomial representation.

For each approximant of a given different order and with a given number of control parameters, we obtain a prediction for the position of the singularity obtained from the condition of zero crossing of the inverse approximant. Let us compare these values between them and with the critical time derived from the corresponding initial polynomial fit. We find the following results:

$$t_{c21}=1.566\,45,$$
$$t_{c31}=1.551\,34,\quad t_{c32}=1.570\,67,$$
$$t_{c41}=1.551\,93,\quad t_{c42}=1.570\,48,\quad t_{c43}=1.570\,79,$$

corresponding to the following multipliers defined by Eq. (12):

$$M_{21}(t_{c21})=0.6484,$$
$$M_{31}(t_{c31})=0.689\,55,\quad M_{32}(t_{c32})=0.637\,08,$$
$$M_{41}(t_{c41})=0.687\,43,\quad M_{42}(t_{c42})=0.637\,72,$$
$$M_{43}(t_{c43})=0.636\,63.$$

The values t_{c21}, t_{c31}, t_{c32}, t_{c41}, t_{c42}, and t_{c43} should be compared with the values determined from the "bare" polynomials

$$t_{20}=1.592\,45,\quad t_{30}=1.569\,91,\quad t_{40}=1.570\,82.$$

The convergence of this last sequence can be analyzed by looking at the corresponding sequence of multipliers $m_i(t,0)$ defined by Eq. (5),

$$m_2(t_{02})=0.577\,35,\quad m_3(t_{03})=0.639\,85,$$
$$m_4(t_{04})=0.636\,51.$$

Overall these results show indeed a good convergence for t_{c41}, t_{c42} and t_{c43} by increasing the number of control parameters and along the diagonal $t_{c21}, t_{c32}, t_{c43}$ by increasing simultaneously both the order of the polynomial and the number of control parameters. The same observation holds for the multipliers M. In contrast, the bare polynomials give a slower nonmonotonous convergence. This confirms the improvement brought by our scheme to obtain a better determination of the critical time t_c.

We note also that the best estimate $t_c=1.570\,78$ obtained from the method of Padé approximant is inferior to our best estimate.

B. Examples where relaxing the accuracy-through-order relation may improve the performance of the approximants

As we discussed, the accuracy-through-order relationship holds for the approximant $\phi_{nn-1}^*(t)$, meaning simply that

$$\phi(t)-\phi_{nn-1}^*(t)\simeq O(t^{1+n}),\quad t\to 0.$$

For all other approximants, it holds in a weaker form, e.g., $\phi(t)-\phi_{nn-2}^*(t)\simeq O(t^n)$, $t\to 0$. We now would like to demonstrate the possibility that suppressing (allowing) some error in the limit of $t\to 0$ does not necessarily minimize (increase) the error for the sought "nonsmall" t values.

Consider the simple and generic function $\phi(t)=(1-t)^{5/4}$ exhibiting a finite-time singularity (in its second-order derivative) at $t_c=1$. The coefficients of its polynomial expansion for small t are

$$a_0=1,\quad a_1=-1.25,\quad a_2=0.156,\quad a_3=0.039,$$
$$a_4=0.017.$$

Using the same approach as above, we find the following predictions for the critical time:

$$t_{c21}=0.894\,66,$$
$$t_{c31}=0.925\,59,\quad t_{c32}=0.935\,71,$$
$$t_{c41}=0.947\,88,\quad t_{c42}=0.970\,05,\quad t_{c43}=0.956\,63;$$

corresponding to the multipliers

$$M_{21}(t_{c21})=0.794\,19,$$
$$M_{31}(t_{c31})=0.709\,11,\quad M_{32}(t_{c32})=0.681\,83,$$
$$M_{41}(t_{c41})=0.628\,18,\quad M_{42}(t_{c42})=0.560\,84,$$
$$M_{43}(t_{c43})=0.608\,73.$$

In constrast, the critical times determined from the zeros of the "bare" polynomials are as follows:

$$t_{20}=0.901\,61,\quad t_{30}=0.934\,74,\quad t_{40}=0.951\,18,$$

with the following multipliers

$$m_2(t_{02})=0.7746,\quad m_3(t_{03})=0.6844,\quad m_4(t_{04})=0.630\,33.$$

The value of t_{c42}, obtained from the approximant that has the smallest multiplier $M_{42}(t_{c42})$ and thus the most stable solution, gives the best estimate of the critical time t_c. This is notwithstanding the fact that the corresponding approximant does not satisfy the strict accuracy-through-order relationship. The approximant that does satisfy the accuracy-through-order relationship gives an estimate t_{c43} that is further away from the true value. We also note that the best estimate provided by Padé approximants technique applied to the expansion of $\phi(t)=(1-t)^{5/4}$ up to the fourth order gives the prediction $t_c=0.965$, which is also inferior compared to our best estimate.

Let us consider another generic function $\phi(t)=(1-t)^{3/4}$ which has now a finite-time-singularity in its first derivative at $t_c=1$. The coefficients of its polynomial expansion for small t are

$$a_0=1, \quad a_1=-0.75, \quad a_2=-0.094, \quad a_3=-0.039,$$

$$a_4=-0.022.$$

In this case, we find that all multipliers are larger than 1: the convergence of the procedure cannot be guaranteed or expected *a priori*.

Nevertheless, even in such extremely unstable case, the analysis of the table of predicted critical times turns out to be useful:

$$t_{c21}=1.1542,$$

$$t_{c31}=1.079\,17, \quad t_{c32}=1.090\,91,$$

$$t_{c41}=1.017\,94, \quad t_{c42}=1.0377, \quad t_{c43}=1.060\,62.$$

Note that the value t_{c41} (again obtained from the approximant that does not satisfy accuracy-through-order relationship) appears in hindsight to be the best estimate. It is also much better than values determined from the zeros of the "bare" polynomials,

$$t_{20}=1.163\,98, \quad t_{30}=1.1087, \quad t_{40}=1.081\,29.$$

The best estimate $t_c=1.047$ of the technique of Padé approximants is inferior to our best estimate. The problem with this example however is that we cannot rely on the multipliers to guide us in choosing which prediction should be preferred.

V. NONLOCAL CONTROL THROUGH THE MOMENTS OF THE FUNCTION TO PREDICT

Section II and our subsequent tests have shown that the control parameters provide a mean to improve the extrapolation to the future by imposing some constraint on the approximants. Up to now, we have used the constraint that the approximants must retrieve the "bare" polynomial expansions at small times t. This corresponds to constraints that are local in time.

It is interesting and potentially useful to investigate the possibility of using more global "nonperturbative" constraints. A possible example is when, either from *a priori* theoretical knowledge or from experimental or empirical measurements, we get hold of the first $j-1$ moments μ_i, $i=1,2,\ldots,j-1$ of the sought function $\phi(t)$ in some interval $[0,T]$,

$$\mu_i=\int_0^T t^{i-1}\phi(t)dt. \qquad (37)$$

Our notation means that, for $j=2$, we know only the zero moment [mass or integral $\phi(t)$ from 0 to T], for $j=3$, we know both the zeroth and first moment, etc.

Endowed with this knowledge of the first $j-1$ moments, we can condition the control parameters $\tau_1,\tau_2,\ldots,\tau_{j-1}$ demanding that the reconstructed approximants have exactly the right values of their moments,

$$\int_0^T \phi_j^*(t,\tau_1,\tau_2,\ldots,\tau_{j-1})t^{i-1}dt=\mu_i. \qquad (38)$$

For $j=2$, we have one equation for τ_1,

$$\int_0^T \phi_2^*(t,\tau_1)dt=\mu_0.$$

For $j=3$, we obtain two equations for τ_1 and τ_2,

$$\int_0^T \phi_3^*(t,\tau_1,\tau_2)dt=\mu_0, \quad \int_0^T \phi_3^*(t,\tau_1,\tau_2)tdt=\mu_1.$$

For $j=4$, we have three equations for τ_1, τ_2, and τ_3,

$$\int_0^T \phi_4^*(t,\tau_1,\tau_2,\tau_3)dt=\mu_0, \quad \int_0^T \phi_4^*(t,\tau_1,\tau_2,\tau_3)tdt=\mu_1,$$

$$\int_0^T \phi_4^*(t,\tau_1,\tau_2,\tau_3)t^2dt=\mu_2.$$

Based on these conditions, two different problems seem most natural. The first one is to construct an approximate representation of the function $\phi(t)$ in the same interval $[0,T]$ where moments are given or measured. In the case where the moments are obtained through some experimental procedure leading to some measurement errors, this first problem amounts to filter out the noise in the measurement interval $[0,T]$. The second problem that we shall address here consists in extrapolating to times $t>T$. The time T usually corresponds to the last available measurement on the system history. The time horizon t for the "prediction" depends on the specificity of the system and is usually taken proportional to the sampling time between two measurements, that is, $t-T\ll T$. Actual calculations are often performed in a "moving window."

Using the previous example of the function $1/\cos(t)$, let us consider the following case $T=\sqrt{2}$. This value is "natural" as it is the root of $1+b_1t^2$. Constraining the control parameters by the knowledge of the moments in the interval $[0,\sqrt{2}]$, we obtain the corresponding approximants. The analyses of the zero of the inverse approximants give the following estimations for the critical times and the corresponding multipliers:

$$t_{c21}=1.568\,88,\quad t_{c32}=1.570\,77,\quad t_{c43}=1.570\,796,$$

$$M_{21}(t_{c21})=0.643\,88,\quad M_{32}(t_{c32})=0.636\,78,$$

$$M_{43}(t_{c43})=0.636\,62.$$

Notice the extremely good quality of the convergence of both the critical times and of the multipliers.

It is also possible to use a hybrid approach, where some control parameters are obtained from the agreement with the polynomial expansion at small time t, while the remaining ones are determined from the conditions on the known moments. As an illustration, we show the fifth-order approximant

$$\phi_{5I}^{*}(t,\tau_1,\tau_2,\tau_3,\tau_4)=\left[1+b_1t^2\exp\left(\frac{b_2}{b_1}\tau_1t^2\exp\left\{\frac{b_3}{b_2}\tau_2t^2\times\exp\left[\frac{b_4}{b_3}\tau_3t^2\exp\left(\frac{b_5}{b_4}\tau_4t^2\right)\right]\right\}\right)\right], \tag{39}$$

where $\tau_1=1$ is conditioned by the polynomial expansion and the other control parameters should be calculated from the system of equations

$$\int_0^T\phi_{5I}^{*}(t,1,\tau_2,\tau_3,\tau_4)dt=\mu_0,$$

$$\int_0^T\phi_{5I}^{*}(t,1,\tau_2,\tau_3,\tau_4)t\,dt=\mu_1,$$

$$\int_0^T\phi_{5I}^{*}(t,1,\tau_2,\tau_3,\tau_4)t^2dt=\mu_2. \tag{40}$$

This system can only be solved numerically.

Note that we do not even need to know the exact values b_3, b_4, and b_5 of the polynomial expansion since they can be included in the corresponding controls. This results from the fact that the constraints on the moments overwhelm the initial information on the coefficients of the polynomial expansion. We find $t_{c54}=1.570\,796$ in extremely good agreement with the exact critical time $t_c=\pi/2=1.570\,796\,3$.

VI. CLASSIFICATION AND FORECASTING OF CRITICAL TIMES BEYOND MEAN FIELD

We now use the formalism of Sec. II and relax the condition $s\to+\infty$ on the exponent of the algebraic transformation, which amounted to impose the mean-field value $z=-1$ of the critical exponent. The control exponent s will be determined from the optimization of the convergence and the stability of the renormalization flow according the general principles developed by Yukalov and Gluzman [23–30].

A. General procedure

Consider, as before, an expansion of an observable $\phi(t)$ in powers of a variable t (time) given by $\phi_k(t)=\Sigma_{n=0}^{k}a_nt^n$, where $a_0=1$ without loss of generality by suitable normalization. The method of algebraic self-similar renormalization [23–25] gives the following general recurrence formula for the approximant of order k as a function of the expansion $\phi_{k-1}(t)$ up to order $k-1$:

$$\phi_k^{*}(t)=\phi_{k-1}(t)\left[1-\frac{ka_k}{s}t^k\phi_{k-1}^{k/s}(t)\right]^{-s/k}\equiv\left[\phi_{k-1}^{-k/s}(t)-\frac{ka_k}{s}t^k\right]^{-s/k}, \tag{41}$$

where, in general, $s=s_k(t)$ depends on the approximation number k and on the variable t. These approximants automatically agree with their corresponding polynomial expansions and the sole way to impose some control is to restrict s using some conditions of rather general nature such as convergence of the sequence of approximants.

In the sequel, we assume that only the second-order expansion is available. In order to determine the critical exponent z, we follow Yukalov and Gluzman [25] and construct the two approximants available from the knowledge of the two coefficients a_1 and a_2. They can be readily obtained from the general formula (41). The first-order approximant is simply

$$\Psi_1^{*}(t)=\left(1-\frac{a_1}{s_1}t\right)^{-s_1}. \tag{42}$$

Representing $\phi_2(t)$ as $\phi_2(t)=1+a_1t[1+(a_2/a_1)t)$ and applying the general formula to the expression in brackets, we obtain the second-order approximant

$$\Psi_2^{*}(t)=1+a_1t\left(1-\frac{a_2}{a_1s_2}t\right)^{-s_2}. \tag{43}$$

Let us assume further that $s_1=s_2=s$, where s is the limiting value of the control function of the algebraic transformation at the critical point. It is apparent from the form of Eq. (41) that s plays the role of the critical index z. As it was explained in Ref. [25], this is justified in the vicinity of a stable fixed point.

The condition of maximum stability of the renormalization flow is equivalent to imposing that the difference $\Psi_2^{*}-\Psi_1^{*}$ be a minimum with respect to the set of parameters (the so-called minimal difference condition). We discuss below an application of the technique applied to direct second-order expansions.

In the present work, we are interested in testing for the existence of a finite-time singularity or critical point. Looking for such an occurrence, we need to solve the minimal difference condition which amounts to look for the solutions of the two equations in terms of the two variables t_c and s,

$$\Psi_1^{*}(t_c,s)=0\quad\text{and}\quad\Psi_2^{*}(t_c,s)=0. \tag{44}$$

The vanishing of Ψ_1^{*} given by Eq. (42) gives

$$t_c=s/a_1. \tag{45}$$

The second condition $\Psi_2^*=0$ with Eq. (43) provides an estimate s for the critical index. In terms of Froude parameter F_d defined by Eq. (18), the second condition of Eq. (44) can be conveniently written as

$$1+s(1-F_d^{-1})^{-s}=0. \tag{46}$$

This gives $s=s(F_d)$ as a function of Froude number. In the cases when this equation does not have a real solution, we determine the control parameter s (which, we recall, is a more general entity than the critical index) from the minimization of $\Psi_2^*(t_c,s)$,

$$\min_s[1+s(1-F_d^{-1})^{-s}]. \tag{47}$$

The Yukalov-Gluzman technique then confronts the two approximants Ψ_1^* and Ψ_2^*: after their difference is minimized, it remains to decide which one of them is the best resummed expression originating from the original perturbative expansion. Implicit in this approach is the concept that the renormalization approach might not be fully convergent asymptotically but only locally. Such a decision can be made based on the analysis of the corresponding multipliers

$$M_j(t,s)\equiv\left[\frac{\partial\Psi_j^*(\varphi,s)}{\partial\varphi}\right]_{\varphi=P_1(t,0)}, \quad j=1,2,$$

yielding

$$M_1(t,s)=\left(1-\frac{a_1 t}{s}\right)^{-(1+s)}, \tag{48}$$

$$M_2(t,s)=\left(1-\frac{a_2}{a_1 s}\ t\right)^{-s}\left[1+\frac{a_2}{a_1}t\left(1-\frac{a_2}{a_1 s}\ t\right)^{-1}\right]. \tag{49}$$

The most stable solution corresponding to the smallest $|M_j|$ should then be selected. We find, in general, that Ψ_1^* has the smallest multiplier in the critical region, which onset is determined by the condition $|M_1(t,s)|\leqslant|M_2(t,s)|$, provided that a solution to Eq. (46) or Eq. (47) exists. On the other hand, we find that Ψ_2^* prevails in some "pseudocritical" regime when the first solution Ψ_1^* becomes unstable. One can make this selection process automatic by the weighting procedure advocated in Ref. [30] which has also been used in Ref. [32]. The weighting procedure amounts to defining an average of the two approximants with weights inversely proportional to their multipliers. The rational for this approach is that the inverse of the multipliers can be shown to play a role similar to the probability that the system visits the dynamical state described by the corresponding approximant. The resulting function is

$$\Psi^*(t,s)=\frac{\Psi_1^*(t,s)|M_1(t,s)|^{-1}+\Psi_2^*(t,s)|M_2(t,s)|^{-1}}{|M_1(t,s)|^{-1}+|M_2(t,s)|^{-1}}. \tag{50}$$

Since we are interested in the physically meaningful case when the critical time $t_c=s/a_1$ is positive (it may be infinite), a modification is required when s/a_1 is found negative or simply when there is no real solution to Eq. (44). We take this situation as a signal that one should use the inverse function defined by Eq. (20) as the relevant expansion to obtain the most stable scenario. The sought function is then defined as the inverse of the weighted average for inverse renormalized approximant. The approximants and corresponding multipliers are calculated using the parameters a_i's [Eq. (17)] changed into b_i's [Eq. (20)]. The final solution reads

$$\begin{aligned}\Psi^*(t,s) &\equiv \Psi_I^*(t,s)^{-1}\\ &=\left(\frac{\Psi_{1I}^*(t,s)|M_{1I}(t,s)|^{-1}+\Psi_{2I}^*(t,s)|M_{2I}(t,s)|^{-1}}{|M_{1I}(t,s)|^{-1}+|M_{2I}(t,s)|^{-1}}\right)^{-1}.\end{aligned} \tag{51}$$

B. Negative velocity $a_1<0$ (downward trend)

Using the general procedure of Sec. VI A, we now present the corresponding classification of the different possible regimes.

1. Negative acceleration $a_2<0$

In this situation, the inverse function has always a singularity. This corresponds for the direct observable function to vanish in finite time (critical regime I) at $t_c=s(F_d)/a_1$, where $s(F_d)$ is the negative solution of Eq. (46). Both approximants Ψ_1^* and Ψ_2^* contribute to the expression (51), and Ψ_2^* progressively dominates as time approaches t_c.

Solution Ψ_2^* contribute to the average (50) more than Ψ_1^*, because $|M_2|$ is always smaller than $|M_1|$. As the multiplier of $\Psi_1^*(t)$ eventually blows up to infinity at t_c, $\Psi_2^*(t)$ ends by dominating the behavior of $\Psi^*(t)$.

The asymptotic behavior of the average close to t_c is determined by Ψ_2^* and is characterized by exponent $z=1$, notwithstanding the fact that the control exponent s is fractional. This corresponds to the situation where the observable goes to zero linearly in time.

2. Positive moderate acceleration $0<a_2<a_1^2/F_0$, $F_0<F_d$ where $F_0=(1-e^{-1/e})^{-1}$

In this region of parameter $F_0=(1-e^{-1/e})^{-1}=3.249<F_d$, the observable still goes to zero in finite time (critical regime I). The time t_c at which the observable vanishes is given by the same formula as in Sec. VI B 1. This solution exists as long as there is a solution $s(F_d)$ to Eq. (46). When F_d becomes too small, Eq. (46) does not possess a solution and this corresponds to the critical regime II discussed in the following section. The boundary between these two regimes occurs at the Froude value F_0 determined by adding the condition

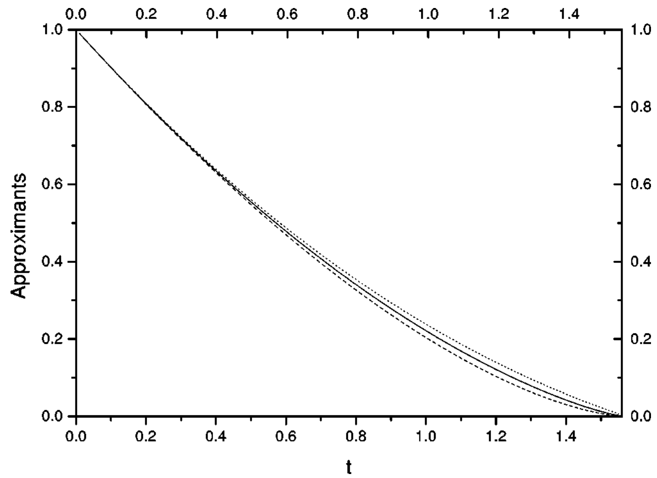

FIG. 1. First-order approximant Ψ_1^* [Eq. (42)], dashed line), second-order approximant Ψ_2^* [Eq. (43), dotted line] and their weighted average given by Eq. (50) (continuous line) as a function of time, for positive moderate acceleration $0<a_2<a_1^2/F_0$, $F_0<F_d$, where $F_0=(1-e^{-1/e})^{-1}$. The approximants and time variables are dimensionless.

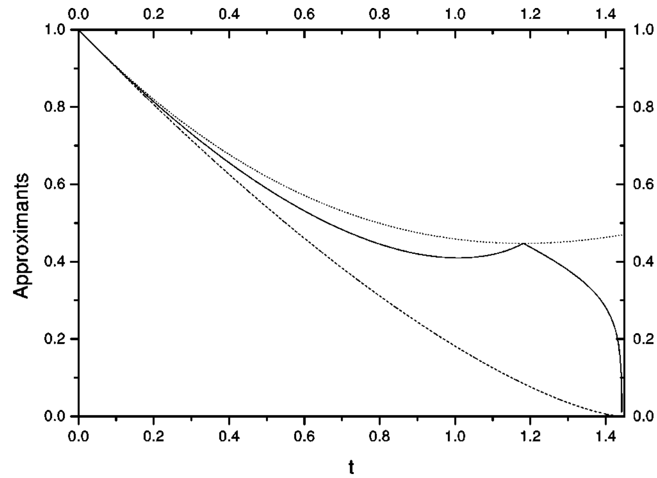

FIG. 2. Dependence of the two approximants $\Psi_1^*(t)$ [Eq. (42), dashed line] and $\Psi_2^*(t)$ [Eq. (43), dotted line] and of their weighted average $\Psi^*(t)$ given by Eq. (50) (continuous line) in the regime of strong positive acceleration $a_1^2/F_0<a_2$, $F_d<F_0$. The approximants and time variables are dimensionless.

$$\frac{\partial(\Psi_2^*(s/a_1,s))}{\partial s}=0 \tag{52}$$

to the general equation $\Psi_2^*(s/a_1,s)=0$. The minimum, solution of Eq. (52), is located at

$$s_{\min}=\frac{1}{\ln\left(\frac{F_d-1}{F_d}\right)}, \tag{53}$$

and coincides with the zero of Ψ_2^* only for the specific value of the Froude number F_0 thus determined by

$$F_0=(1-e^{-1/e})^{-1}=3.249. \tag{54}$$

As $F_d\to\infty$, Eq. (46) can be solved exactly and $s(F_d\to\infty)=-1$, which gives the mean-field value $z=1$. In other words, in the case of $F_d\to\infty$, corresponding to a linear function $\phi(t)=1-|a_1|t$, when confronted with its expansion, our technique will reconstruct $\phi(t)$ exactly. Let us expand Eq. (35) around this exactly solvable limit in powers of a small parameter $1/F_d=y$,

$$1+s(1-F_d^{-1})^{-s}\simeq 1+s+s^2y+\cdots. \tag{55}$$

Then,

$$s=\frac{1}{2y}(-1+\sqrt{1-4y})\simeq -1-y+\cdots(y\to 0). \tag{56}$$

This expression breaks down around $F_d=4$, and the numerical solution to the Eq. (46) should be considered in the region of Froude parameter $F\leqslant F_0$. Note that $s(F_0)=-e$.

Solution Ψ_1^* starts to contribute to the average (50) more than Ψ_2^*, as soon as t satisfy condition $|M_1|\ll|M_2|$, as shown in Fig. 1, which represents the dependence of the two approximants $\Psi_1^*(t)$ [Eq. (42)] and $\Psi_2^*(t)$ [Eq. (43)] and of their weighted average $\Psi^*(t)$ given by Eq. (50). As the multiplier of $\Psi_1^*(t)$ eventually vanishes at t_c, $\Psi_1^*(t)$ ends by dominating the behavior of $\Psi^*(t)$ and the average demonstrates critical behavior with positive fractional exponent $z=-s(F_d)$. Thus, in this region of F_d, as t goes to t_c, we obtain a critical behavior with fractional s playing the role of critical index z. It means that the exponent is now different from -1 and is determined by the solution of Eq. (46).

3. Strong positive acceleration $a_1^2/F_0<a_2$, $F_d<F_0$

The critical regime (I) is now replaced by the critical regime (II), such that Eq. (46) does not possess a solution and the control exponent and critical time are determined from the minimization of Eq. (47), which gives

$$s=s_{\min}(F_d)=\frac{1}{\ln\left(\frac{F_d-1}{F_d}\right)}. \tag{57}$$

The critical index is $z=-s_{\min}(F_d)$, leading to a logarithmic correction to the mean-field value. The critical time is given by

$$t_c=\frac{s_{\min}(F_d)}{a_1}. \tag{58}$$

Figure 2 shows the dependence of the two approximants $\Psi_1^*(t)$ [Eq. (42)] and $\Psi_2^*(t)$ [Eq. (43)] and of their weighted average $\Psi^*(t)$ given by Eq. (50). The characteristic feature is the existence of a minimum at time $t_{\min}$ of $\Psi_2^*(t)$ and, therefore of a nonmonotonous behavior also of the average $\Psi^*(t)$, with $t_{\min}$ given by

$$t_{\min}=-\frac{a_1}{a_2}\frac{s_{\min}}{s_{\min}-1}, \tag{59}$$

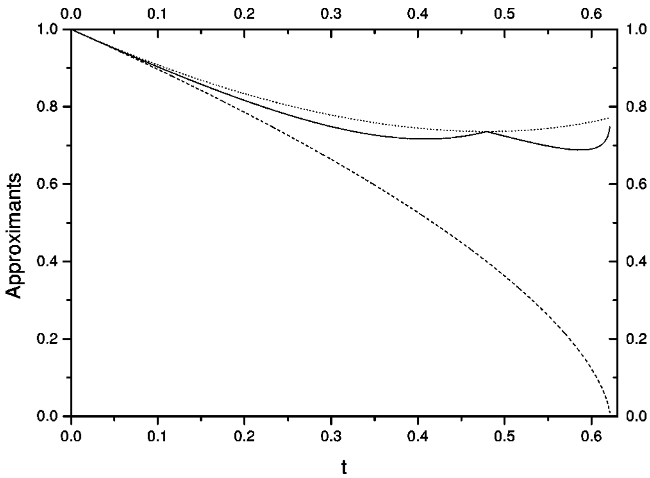

FIG. 3. Same as Fig. 2 in the regime $1<F_d<F_{01}$ (pseudocritical regime I). Note that the weighted average $\Psi^*(t)$ exhibits a fast change of direction to reach $\Psi_2^*(t)$ at t_c.

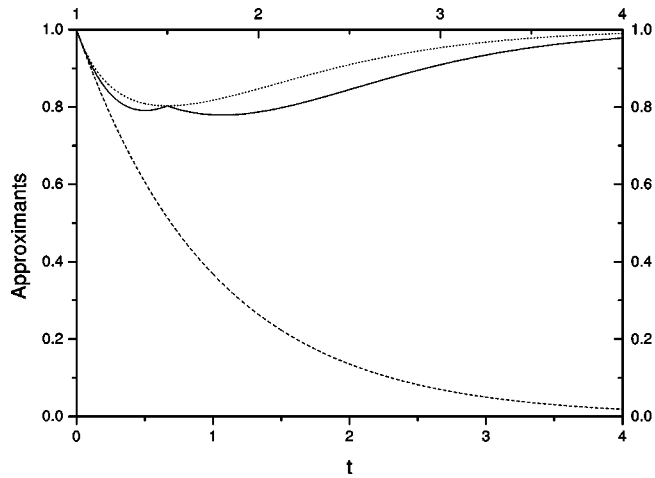

FIG. 4. Same as Fig. 2 for $F_d<1$.

corrected by the ratio $s_{\min}/(s_{\min}-1)$ compared to the mean-field result (25) of Sec. III B 3. The value of $\Psi_2^*(t)$ at $t_{\min}$ is

$$\Psi^*_{2\min}=1-F_d[1+L(F_d)]^{-[1+L(F_d)]/L(F_d)},$$

$$L(F_d)=\ln\left(\frac{F_d}{F_d-1}\right).$$

The trajectory $\Psi^*(t)$ shown in Fig. 2 is rather unusual since after spending some time close to the local minimum of a noncritical branch $\Psi_2^*(t)$, the system suddenly breaks down towards the "critical" branch $\Psi_1^*(t)$, which then ends at a critical point t_c. This means that the critical behavior with exponent $z=-s_{\min}(F_d)$ has not disappeared yet. The drop occurs at a crossover time $t=t_{cros}$ defined as the solution to the equation $|M_1|\approx|M_2|$, with a magnitude $\Delta=\Psi_2^*(t_{cros})-\Psi_1^*(t_{cros})$. This regime is found for the Froude interval $F_{01}<F_d<F_0$, where $F_{01}=(1-e^{-1})^{-1}=1.582$ is the solution of the equation

$$1+\frac{1}{\ln\left(\frac{F_{01}-1}{F_{01}}\right)}=0, \tag{60}$$

corresponding to the Froude value at which the multiplier $M_1(t,s)$ changes from stable ($M_1<1$) to unstable ($M_1>1$) behavior. As the regime $1<F_d<F_{01}$ (pseudocritical regime I) sets in, the multiplier $M_1(t,s)$ becomes larger than 1, increases with time and diverges at t_c. Rather than converging to the $\Psi_1^*(t)$ approximant, the weighted average $\Psi^*(t)$ exhibits a fast change of direction to reach $\Psi_2^*(t)$ at t_c, as shown in Fig. 3. The critical branch has disappeared as the noncritical branch Ψ_2^* dominates. The presence of the approximant scenario Ψ_1^* is felt only in the existence of some oscillations accompanying the Ψ_2^* scenario. This regime exists for $1<F_d<F_{01}$.

For $F_d<1$ (pseudocritical regime II), the minimum of Eq. (47) disappears and there is no solution either to Eq. (46) or to (47) anymore. This implies that we should use the inverse expansion in terms of the coefficients $b_1>0,b_2<0(F_I<0)$. The corresponding control exponent s is then obtained from the condition

$$1+s(1-F_I^{-1})^{-s}=0, \tag{61}$$

which has a negative solution $s=s(F_I)$ leading to $t_c=s/b_1<0$, which is not allowed. However, similarly to the previous strategy to replace Eq. (46) by Eq. (47), we can look for the solution that minimizes the left-hand side of Eq. (61), which gives $s\to\infty$. This corresponds to a pseudocritical regime which is reminiscent of the last phase of the previous regime, but with formally infinite t_c. In the limit $s\to\infty$, we obtain

$$\Psi_{1I}^*(t)=\exp(b_1t), \quad M_{1I}(t)=\exp(b_1t),$$

$$\Psi_{2I}^*(t)=1+b_1t\exp\left(\frac{b_2}{b_1}t\right), \quad M_{2I}(t)=\left(1+\frac{b_2}{b_1}t\right)\exp\left(\frac{b_2}{b_1}t\right).$$

Note that $\Psi_{2I}^*(t)$ is qualitatively similar to the mean-field solution of Sec. III, derived for the same region of parameters. The multiplier $M_{1I}(t)$ is always larger than 1, which implies that the scenario Ψ_{2I}^* always dominates in the weighted average, although the contribution of Ψ_{1I}^* is responsible for the existence of an extra minimum in the trajectory of Ψ^* given by expression (51), as shown in Fig. 4.

C. Positive velocity $a_1>0$ (upward trend)

1. Negative acceleration: $a_2<0$

One can still define the control exponent s from the condition (46), but t_c becomes negative which is undesirable. As in the preceding section, we turn to the next possibility which is to use Eq. (47), whose only solution is $s\to+\infty$. This solution turns out to minimize the difference between the two approximant scenarios. This regime is the inversion of the pseudocritical regime II just presented above (only with $F_d<0$ instead of F_I), as it is described by the following solutions,

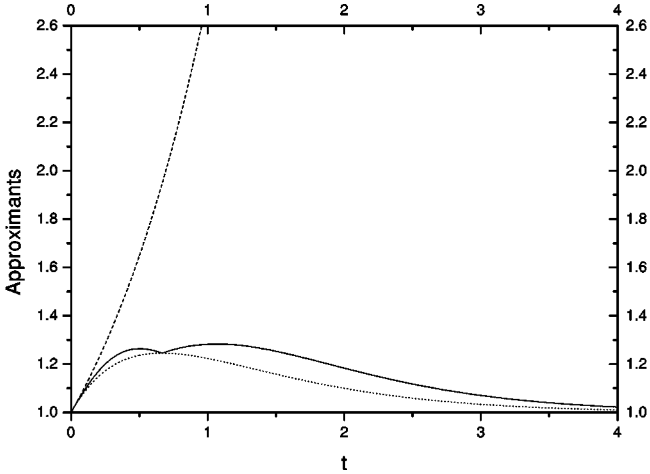

FIG. 5. Same as Fig. 2 for a negative acceleration $a_2<0$.

$$\Psi_1^*(t)=\exp(a_1 t), \quad M_1(t)=\exp(a_1 t). \tag{62}$$

$$\Psi_2^*(t)=1+a_1 t\exp\left(\frac{a_2}{a_1}t\right), \quad M_2(t)=\left(1+\frac{a_2}{a_1}t\right)\exp\left(\frac{a_2}{a_1}t\right). \tag{63}$$

Note that $\Psi_2^*(t)$ is nothing but the mean-field solution of Sec. III, derived for the same parameter region. The multiplier $M_1(t)$ is found to be always larger than 1. Therefore, Ψ_2^* dominates in the weighted average trajectory Ψ^* Eq. (50). The contribution from Ψ_1^* induces splitting of the mean-field maximum (at $t=-a_1/a_2$) as shown in Fig. 5.

2. Moderate positive acceleration: $0<a_2<(F_0-1)/F_0 a_1^2$; $F_0/(F_0-1)<F_d$

In this case, although Eq. (46) has a solution for $s<0$, the corresponding t_c is negative. After inversion of the initial series, this region of parameters is equivalent to $1<F_I<F_0$. This regime corresponds to the inverse of the critical regime II described above and can be described similarly.

Consider first the region of $F_{01}<F_I<F_0$. There is a minimum of the curve Ψ_{2I}^* and the shape of the observable $(\Psi_I^*)^{-1}$ [see Eq. (51)] is significantly nonmonotonous, due to contribution from Ψ_{2I}^*. The time $t_{\min}$ of the minimum of $(\Psi_{2I}^*)^{-1}$ (maximum of Ψ_{2I}^*) is given by

$$t_{\min}=-\frac{b_1}{b_2}\frac{s_{\min}}{s_{\min}-1}, \tag{64}$$

where

$$s_{\min}=\frac{1}{\ln\left(\frac{F_I-1}{F_I}\right)} \tag{65}$$

is the solution of the minimization

$$\min_s[1+s(1-F_I^{-1})^{-s}].$$

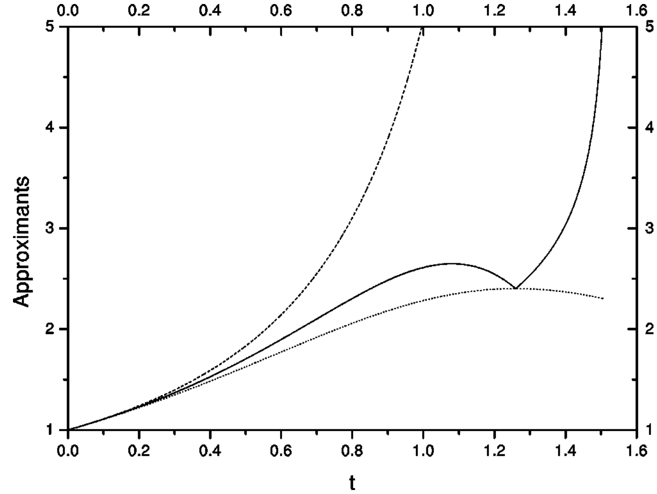

FIG. 6. Same as Fig. 2 in the regime of moderate positive acceleration: $0<a_2<(F_0-1)/F_0\ a_1^2$; $F_0/(F_0-1)<F_d$.

The maximum value is $(\Psi_{2I(\min)}^*)^{-1}$ where

$$\Psi_{2I(\min)}^*=\{1-F_I[1+L(F_I)]^{-[1+L(F_I)]/L(F_I)}\},$$

$$L(F_I)=\ln\left(\frac{F_I}{F_I-1}\right). \tag{66}$$

The trajectory has the same topology as shown in Fig. 2 except that it is the inverse of the function shown in Fig. 2. The observable is predicted as the weighted average scenario given by Eq. (51) and is shown in Fig. 6. The "critical" branch Ψ_{1I}^* shapes the weighted average Ψ_I^* as t_c is approached. The weighted average scenario goes to infinity in finite time $t_c=s_{\min}/b_1$, with negative $z=s_{\min}$ describing the power-low divergence. In terms of the coefficients a_i of the polynomial expansion, this regime holds for $(F_{01}-1)/F_{01}a_1^2<a_2<(F_0-1)/F_0 a_1^2$, i.e., for $F_0/(F_0-1)<F_d<F_{01}/(F_{01}-1)[(F_{01}-1)/F_{01}=0.368$, $(F_0-1)/F_0=0.692]$. These conditions are equivalent to $F_{01}<F_I<F_0$.

As F_I becomes smaller than F_{01}, the multiplier $M_{1I}(t,s)$ changes from stable ($M_{1I}<1$) to unstable ($M_{1I}>1$). As a consequence and similarly to the change from Fig. 2 to Fig. 3, the weighted average scenario changes considerably and does not exhibit a critical divergence at t_c anymore. The scenario Ψ_{1I}^* is felt only in the creation of a few oscillations around Ψ_{2I}^*. This regime holds for $1<F_I<F_{01}$ and mirrors the pseudocritical regime I previously described. In terms of initial coefficients, it corresponds to $0<a_2<(F_{01})-1/F_{01}a_1^2$.

3. Large positive acceleration: $(F_0-1)/F_0 a_1^2<a_2$, $F_I>F_0$

In terms of the inverse Froude number, this regime holds for $F_I>F_0$. The observable goes to infinity in finite time at a critical time t_c, which is determined from the condition that the inverse quantities cross zero. The corresponding control exponent $s(F_I)$ is the solution of

$$1+s(1-F_I^{-1})^{-s}=0, \tag{67}$$

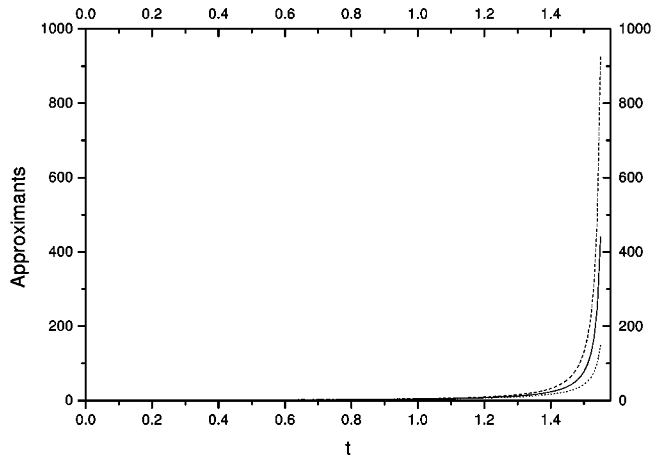

FIG. 7. Same as Fig. 2 in the regime $(F_0-1)/F_0 a_1^2 < a_2$, $F_I > F_0$.

and

$$t_c = s(F_I)/b_1 .$$

The existence of the finite-time singularity holds as long as $F_I > F_0$. This condition can be reexpressed in terms of the direct Froude number and gives $F_d < F_0/(F_0-1) = 1.445$. This regime mirrors the critical regime I. Given a second-order expansion $\phi_2(t) \simeq 1 + |a_1|t + |a_1|^2 t^2$ of a simple pole $\phi(t) = (1-|a_1|t)^{-1}$ (with $F_I \to \infty$), our technique will indeed reconstruct it. The weighted average scenario $\Psi_I^*(t,s)^{-1}$ goes to infinity in finite time t_c, with negative $z = s(F_I)$ describing the power-law divergence. In Fig. 7 we demonstrate the dependence of the two approximants $\Psi_1^*(t)$ and $\Psi_2^*(t)$ and of their weighted average $\Psi^*(t)$ given by Eq. (51).

VII. CONCLUDING REMARKS

Starting from a representation of the early time evolution of a dynamical system in terms of the polynomial expression of some observable $\phi(t)$ as a function of time, we have investigated the conditions under which this early time dynamics may or may not lead to a finite-time singularity. The corresponding classification has been performed from the point of view of the functional renormalization method of Yukalov and Gluzman [18–30], with the purpose of identifying the most stable scenarios, given the early time dynamics. The direct extension of this work is to test our predictions empirically, following the methodology of Ref. [16] developed for a particular case.

Our classification of the singular and nonsingular behavior of functions $\phi(t)$ has been performed on the basis of approximations by polynomials of second order in t. This is *a priori* justified by the nice properties of the exponential approximants $\Phi_2^*(t;\tau_1)$ defined in Eq. (9). However, one could doubt the practical usefulness of low-order polynomial approximations if the series converge too slowly. An important question is thus whether the qualitative features of the exponential approximants $\Phi_2^*(t;\tau_1)$ obtained here could not be changed completely by including a larger number of coefficients a_j of the defining time series (1), i.e., by considering instead more complicated approximants $\Phi_n^*(t;\tau_1,\ldots,\tau_{n-1})$ with higher values of n. This question is very difficult and cannot be answered in general without a thorough study of the classifications including higher-order approximants. Such introduction of the higher-order terms in the expansion makes, however, any analytical classification very difficult and unnecessarily redundant. We expect the results obtained in the lowest second order to survive at least in a general sense, because it captures the fundamental competition between velocity (coefficient a_1) and acceleration (coefficient a_2). Higher-order terms act only to renormalize these effective velocities and accelerations when not too far from the critical time t_c. We thus expect that the existence of two broad classes of solutions, with and without finite-time singularity, will hold true for higher-order terms. Let us also point out that, in many realistic problems, one does not have the luxury of more than very few terms obtained by some perturbation theory, In addition, most time series or data are accompanied by noise and only the lowest-order polynomials can be used in data fitting to avoid ill-conditioning and spurious solutions [16]. In such cases, techniques for accelerating the convergence based on a few terms of the expansion become vital.

In addition, we expect our method not to work for all classes of functions. Exponential approximants of the type presented here will give reasonable results when applied to the reconstruction of continuous functions decaying exponentially at infinity. The numerical errors of such reconstructions appear to be much smaller for self-similar exponential approximants than for the standard Padé approximants [36]. On the other hand, nested exponentials become less applicable to the case of functions that give coefficients a_i's growing rapidly in absolute value with their order i. In particular, for functions with coefficients growing as fast as a factorial of their order, Padé approximants outperform superexponentials [34]. This case is particularly important, since it appears, for instance, in the expansions typical to many nonlinear field theories. For instance, the Stieltjes function, $\phi(t) = \int_0^\infty \exp(-u)/(1+tu)du$, which exemplifies such a behavior, has the coefficients of its Euler series [36], diverging as a factorial.

On the other hand, when one is concerned with the calculation of a critical point (or a finite-time singularity as studied here), it is an almost trivial result that exponential approximants will reproduce exactly a power law of the type $(t_c-t)^{-1}$ with exponent -1, based on taking into account any arbitrary orders of the expansion. Therefore, one *a priori* expects that accurate calculations with such technique are possible for the critical-type functions which are not too far from this "mean-field" finite-time singularity. Further increase of accuracy can come at the expense of lifting the mean-field condition. Indeed, by easing the conditions on the control parameter s, one can obtain a more general family of approximants, allowing for power-laws including constants, as in the asymptotic behavior studied in Ref. [35]. Such approximants were first suggested in Ref. [26] (see also Ref. [16]) for concrete applications).

[1] A. Pumir, B. I. Shraiman, and E. D. Siggia, Phys. Rev. A **45**, R5351 (1992).
[2] A. Bhattacharjee and A. C. S. Ng, Phys. Rev. E **52**, 5110 (1995).
[3] N. M. Zubarev, Phys. Lett. A **243**, 128 (1998).
[4] X. Leoncini, L. Kuznetsov, and G. M. Zaslavsky, Physics of Fluids **12**, 1911 (2000).
[5] M. W. Choptuik, Phys. Rev. Lett. **70**, 9 (2000); Prog. Theor. Phys. Suppl. **136**, 353 (1999).
[6] M. Rascle and C. Ziti, J. Math. Biol. **33**, 388 (1995).
[7] H. K. Moffatt, Nature (London) **404**, 833 (2000).
[8] A. A. Stanislavsky and K. Weron, Physica D **156**, 247 (2001).
[9] A. Johansen and D. Sornette, Int. J. Mod. Phys. C **9**, 433 (1998).
[10] A. Johansen and D. Sornette, Eur. Phys. J. B **18**, 163 (2000).
[11] R. C. Hidalgo, F. Kun, and H. J. Herrmann, Phys. Rev. E **64**, 066122 (2001).
[12] A. Johansen, H. Saleur, and D. Sornette, Eur. Phys. J. B **15**, 551 (2000) and references therein.
[13] S. G. Sammis and D. Sornette, Proc. Natl. Acad. Sci. U.S.A. **99**, 2501 (2002).
[14] A. Johansen, D. Sornette, and D. Ledoit, J. Risk **1**, 5 (1999).
[15] A. Johansen and D. Sornette, Eur. Phys. J. B **17**, 319 (2000).
[16] S. Gluzman, J. V. Andersen, and D. Sornette, Computational Seismology **32**, 122 (2001).
[17] C. Ziehmann, L. A. Smith, and J. Kurths, Phys. Lett. A **271**, 237 (2000).
[18] V. I. Yukalov, Phys. Rev. A **42**, 3324 (1990).
[19] V. I. Yukalov, J. Math. Phys. **32**, 1235 (1991).
[20] V. I. Yukalov, J. Math. Phys. **33**, 3994 (1992).
[21] V. I. Yukalov and E. P. Yukalova, Physica A **206**, 553 (1994).
[22] V. I. Yukalov and E. P. Yukalova, Physica A **225**, 336 (1996).
[23] S. Gluzman and V. I. Yukalov, Phys. Rev. E **55**, 3983 (1997).
[24] V. I. Yukalov and S. Gluzman, Phys. Rev. E **55**, 6552 (1997).
[25] V. I. Yukalov and S. Gluzman, Phys. Rev. Lett. **79**, 333 (1997).
[26] V. I. Yukalov and S. Gluzman, Phys. Rev. E **58**, 1359 (1998).
[27] S. Gluzman and V. I. Yukalov, Phys. Rev. E **58**, 4197 (1998).
[28] V. I. Yukalov and S. Gluzman, Physica A **273**, 401 (1999).
[29] V. I. Yukalov, E. P. Yukalova, and S. Gluzman, Phys. Rev. A **58**, 96 (1998).
[30] V. I. Yukalov and S. Gluzman, Int. J. Mod. Phys. B **13**, 1463 (1999).
[31] L. V. Kantorovich and G. P. Akilov, *Functional Analysis in Normed Spaces* (Pergamon, New York, 1964).
[32] J. V. Andersen, S. Gluzman, and D. Sornette, Eur. Phys. J. B **14**, 579 (2000).
[33] S. Gluzman and D. Sornette, Phys. Rev. E **63**, 066129 (2001).
[34] S. Gluzman, D. Sornette, and V. I. Yukalov, cond-mat/0204326.
[35] C. M. Bender and S. Boettcher, J. Math. Phys. **35**, 1914 (1994).
[36] G. A. Baker, Jr. and P. Graves-Morris, *Padé Approximants*, 2nd ed. (Cambridge University Press, Cambridge, 1996).